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Modelling of turbulent scalar transport in homogeneous turbulence

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Abstract

A transport model for the turbulent heat flux is proposed in consideration with the effect of the mean temperature gradient. A rapid term to express the interaction between the mean temperature and velocity gradients is introduced. The linearity in the exact turbulent heat flux equation is retained in the modelled one, too. The effects of Prandtl number and the turbulent Reynolds number are also taken into account. The comparison is made with existing DNS and experimental data of homogeneous turbulence with and without the mean shear rate and the stable stratification. It indicates that the proposed model agrees well with the DNS and experimental data tested. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Turbulent scalar transport; Turbulence model; Homogeneous turbulence; Countergradient heat flow

1. Introduction

The direct numerical simulation (DNS) of turbulence is now widely performed for various types of turbulent flows. Although the DNS of turbulent heat transfer has become widespreading, its applicability is limited to the turbulences with a low Reynolds number and simple geometry. Thus, the modelling of the turbulent heat flux is still of primary importance for the practical calculation of the turbulent heat flux. One of the distinguished features of the DNS is to provide the turbulent modelling with detailed necessary information for each term of the transport equations. Formerly, the proposed turbulence models were tested against the experimental measurements; on the other hand, it is at present more usual to examine the performance of the models in comparison with the DNS data.

The exact transport equation for the turbulent heat flux can be derived as

$$\frac{D\overline{u_i\theta}}{Dt} = P_{i\theta} + G_{i\theta} + T_{i\theta} + V_{i\theta} + \psi_{i\theta} + \phi_{i\theta} - \varepsilon_{i\theta}, \tag{1}$$

where

$P_{i\theta} = -\overline{u_iu_j}\frac{\partial\Theta}{\partial x_j} - \overline{u_j\theta}\frac{\partial U_i}{\partial x_j}$	production
$G_{i\theta} = -\beta g_i\overline{\theta^2}$	buoyancy production
$T_{i\theta} = -\frac{\partial\overline{u_i\theta u_j}}{\partial x_j}$	turbulent diffusion
$V_{i\theta} = \frac{\partial}{\partial x_j}(\nu\overline{\theta\frac{\partial u_i}{\partial x_j}} + \alpha\overline{u_i\frac{\partial\theta}{\partial x_j}})$	molecular diffusion
$\psi_{i\theta} = -\frac{1}{\rho}\frac{\partial\overline{\theta p}}{\partial x_i}$	pressure diffusion
$\phi_{i\theta} = \frac{1}{\rho}\overline{p\frac{\partial\theta}{\partial x_i}}$	pressure temperature-gradient correlation

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Nomenclature

b_{ij}	anisotropic tensor, $=\overline{u_iu_j}/\overline{q^2}-(1/3)\delta_{ij}$	U_i	mean velocity
C_μ	model constant in the eddy diffusivity model	u_i	fluctuating velocity
$C_{1\theta}$	model constant in the slow term	$\overline{u_iu_j}$	Reynolds stress
$C_{1\theta}$, a, $C'_{1\theta}$, a	model constants in the slow term	$\overline{u_i\theta}$	turbulent heat flux
$C_{1\theta}$, b, $C'_{1\theta}$, b	model constants in the slow term	α	thermal diffusivity
$C_{2\theta}$	model constant in the rapid term	α_t	thermal eddy diffusivity coefficient
$C_{3\theta}$	model constant in the buoyancy term	$\alpha_1 \sim \alpha_6$	model constants in the rapid term
g	gravitational acceleration	β	volumetric expansion coefficient
g_i	gravity vector, $=(0, 0, -g)$	$\beta_1 \sim \beta_6$	model constants in the rapid term
k	kinetic energy, $=\overline{u_iu_i}/2=\overline{q^2}/2$	ε	dissipation of the kinetic energy, $=\nu\overline{u_{i,j}u_{i,j}}$
p	fluctuating pressure	ε_θ	dissipation of the temperature variance, $=\alpha\overline{\theta_{,j}\theta_{,j}}$
Pr	Prandtl number, $=\nu/\alpha$	$\varepsilon_{i\theta}$	dissipation of the turbulent heat flux
Pr_t	turbulent Prandtl number, $=\nu_t/\alpha_t$	Θ	mean temperature
Ri	Richardson number	θ	fluctuating temperature
Ri_g	gradient Richardson number, $=\beta g S_\theta/S^2$	ν	molecular viscosity coefficient
Re_t	turbulent Reynolds number, $=k^2/\nu\varepsilon$	ν_t	eddy viscosity coefficient
S	mean velocity gradient	$\pi_{i\theta}$	$=\phi_{i\theta}-\varepsilon_{i\theta}$
S_θ	mean temperature gradient	ρ	density
St	nondimensional time, $=S\cdot t$	$\phi_{i\theta}$	pressure temperature-gradient correlation
t	time	III	third invaliant, $=b_{ij}b_{jk}b_{ki}$

$\varepsilon_{i\theta}=(\nu+\alpha)\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial\theta}{\partial x_j}}$. dissipation

In these terms, the production and the buoyancy production terms can be calculated exactly from the mean profiles, U_i and Θ, and the second-order correlations, $\overline{u_iu_j}$ and $\overline{u_i\theta}$. Rest of the terms requires modelling. Among them, the turbulent, molecular and pressure diffusion terms are, if integrated over a given domain, reduced to the difference of their values at the boundary. Accordingly, the productions, the pressure temperature-gradient correlation (PTG) and the dissipation are the terms which determine the net balance of the turbulent heat flux within the region.

In case of the heterogeneous turbulence such as wall turbulence, none of the terms in Eq. (1) can be neglected. In case of the homogeneous one, on the other hand, only the productions, PTG and the dissipation contribute the balance equation of the turbulent heat flux. In the present study, the homogeneous turbulence with the mean-shear and buoyant productions is adopted to concentrate the attention to these important terms.

The standard procedure to obtain a formal expression for the pressure is to use the Poisson equation and the Green's theorem

$$p=\frac{\rho}{4\pi}\int\left\{\frac{\partial^2}{\partial x_k\partial x_l}(u_ku_l-\overline{u_ku_l})+2\left(\frac{\partial u_l}{\partial x_k}\right)\left(\frac{\partial U_k}{\partial x_l}\right)+\beta g_k\frac{\partial\theta}{\partial x_k}\right\}'\frac{\mathrm{d}V}{\underline{r}},\tag{2}$$

where the prime $'$ denotes the position at $\underline{r}$. The last term in Eq. (2) represents the effect of buoyancy.

With use of the above expression, the PTG term can be obtained as follows.

$$\phi_{i\theta}=\frac{1}{\rho}\overline{P\frac{\partial\theta}{\partial x_i}}$$

$$\phi_{i\theta}=\underbrace{\frac{1}{4\pi}\int\overline{\left(\frac{\partial\theta}{\partial x_i}\right)\left(\frac{\partial^2(u_ju_k)}{\partial x_j\partial x_k}\right)'}\frac{\mathrm{d}V}{\underline{r}}}_{\phi_{i\theta,1}}+\underbrace{\frac{1}{2\pi}\left(\frac{\partial U_j}{\partial x_k}\right)'\int\overline{\left(\frac{\partial\theta}{\partial x_i}\right)\left(\frac{\partial u_k}{\partial x_j}\right)'}\frac{\mathrm{d}V}{\underline{r}}}_{\phi_{i\theta,2}}$$

$$+\underbrace{\frac{\beta g_j}{4\pi}\int\overline{\left(\frac{\partial\theta}{\partial x_i}\right)\left(\frac{\partial\theta}{\partial x_j}\right)'}\frac{\mathrm{d}V}{\underline{r}}}_{\phi_{i\theta,3}}, \tag{3}$$

where $\phi_{i\theta,\,1}$, $\phi_{i\theta,\,2}$ and $\phi_{i\theta,\,3}$ are called as slow, rapid and buoyancy terms, respectively.

Gibson–Launder [1] (GL) proposed the so-called "basic model" for the PTG term:

$$\phi_{i\theta,\,1} = -C_{1\theta}\frac{\varepsilon}{k}\overline{u_i\theta} \tag{4}$$

$$\phi_{i\theta,\,2} = C_{2\theta}\overline{u_j\theta}\frac{\partial U_i}{\partial x_j} \tag{5}$$

$$\phi_{i\theta,\,3} = C_{3\theta}\beta g_i\overline{\theta^2}. \tag{6}$$

Afterwards, this basic model was extended into several directions. Common procedure is to introduce the anisotropy tensor of Reynolds stress b_{ij}:

$$b_{ij} = \frac{\overline{u_iu_j}}{\overline{q^2}} - \frac{1}{3}\delta_{ij}, \tag{7}$$

Craft and Launder [2] (CL) expanded the coefficient in Eqs. (4)–(6) in terms of b_{ij}; the one in the slow term $C_{1\theta}$, up to the second order of b_{ij} and the other in $C_{2\theta}$ up to the third order. Shih and Shabbir [3] (SS) introduced a concept of the realisability but their expression was quite complicated. Rogers et al. [4] made DNS of the turbulent heat flux and investigated the effect of molecular Prandtl number and the turbulent Reynolds number. Shikazono and Kasagi [5] (SK) abandoned the approach to split the PTG into the slow and rapid terms and obtained a new expression for the PTG; it gave good results but was not tensorially correct.

Jones and Musonge [6] (JM) retained a simpler structure but introduced the mean temperature gradient into the PTG term. Although the mean temperature gradient does not appear in Eq. (3) explicitly, they considered that it might affect the PTG implicitly through the coupling between the energy and momentum equations. Thus, they introduced a new term proportional to the mean temperature gradient into the PTG. This approach was adopted by CL, too.

In the Reynolds stress equation, the pressure–strain term acts to compensate the production term, i.e., the production term is not totally effective but only its certain percentage contributes as the production. In the turbulent heat flux equation, the PTG term plays the same role. However, it contains only the mean shear term; thus the mean temperature part in the production term (Eq. (1)) cannot be compensated. This is a reason why the mean temperature gradient should be included in the PTG even though it does not appear in Eq. (3). Thus, in the present study also, the mean temperature gradient is included in the PTG term.

Another aspect of the present study is to retain the linear structure of the transport equation of the turbulent heat flux in the modelled equation, too. That is, each term in Eq. (1), except the buoyancy production term, contains only one scalar quantity. That is, the turbulent heat flux equation is linear with respect to the temperature. The buoyancy term is ignored tentatively in the discussion here. If two temperature fields θ_1 and θ_2 exist, each heat flux satisfies Eq. (1) independently; that is,

$$\frac{\mathrm{D}\overline{u\theta_1}}{\mathrm{D}t} = \text{Eq. (1)} \quad \text{with } \theta = \theta_1 \text{ and } \Theta = \Theta_1 \tag{8}$$

$$\frac{\mathrm{D}\overline{u\theta_2}}{\mathrm{D}t} = \text{Eq. (1)} \quad \text{with } \theta = \theta_2 \text{ and } \Theta = \Theta_2. \tag{9}$$

If these two temperature fields are overlapped, the resultant heat flux

$$\overline{u\theta_3} = \overline{u\theta_1} + \overline{u\theta_2} \tag{10}$$

must be obtained by the linear summation of Eqs. (8) and (9):

$$\frac{\mathrm{D}\overline{u\theta_3}}{\mathrm{D}t} = \frac{\mathrm{D}\overline{u\theta_1}}{\mathrm{D}t} + \frac{\mathrm{D}\overline{u\theta_2}}{\mathrm{D}t}. \tag{11}$$

This implies that the modelled transport equation should also satisfy the above linear summation rule. This is the linearity principle proposed by Pope [7]. Although this is an evidently fundamental principle, it has been ignored in most of the scalar transport models proposed hitherto. That is, the so-called time constant ratio $R = \overline{\theta^2}\varepsilon/(2k\varepsilon_\theta)$, which conflicts with this principle, is often included in the scalar transport model, because it allows introduction of additional terms and brings more freedom in the modelling. Thus, the linearity principle has been often preferred to be abandoned in many scalar transport models.

The another principle to be considered in the modelling of the turbulent heat flux is the realisability. Since the turbulent heat flux is a kind of the cross correlation, it must satisfy the following inequality:

$$\left(\overline{u_{(i)}\theta}\right)^2 \le \overline{u_{(i)}^2}\cdot\overline{\theta^2}, \tag{12}$$

where no summation rule is applied to (i). Some researchers have placed more emphasis on the realisability principle. To satisfy the above relation, the transport equation for $\overline{u_i\theta}$ must contain some information on $\overline{\theta^2}$. This conflicts with the linearity principle. Thus, to the author's knowledge, no model has been proposed to satisfy both the linearity and realisability. The inequality of Eq. (12) is certainly an obvious mathematical prerequisite; however, it is not easily violated if each turbulent quantities are properly predicted. Thus, in the present study, the linearity principle is adopted. The objective of the present work is to propose a turbulent heat flux model which includes the mean temperature gradient effect in the rapid term of PTG and moreover satisfies the linearity principle. The proposed model is examined in reference to the DNS data of the homogeneous turbulence which includes all the budget terms of the turbulent heat flux transport equation except the diffusive terms.

2. Modelling of the PTG term

In the homogeneous turbulence, all the diffusion terms ($T_{i\theta}$, $V_{i\theta}$ and $\psi_{i\theta}$) are zero; thus, the productions ($P_{i\theta}$ and $G_{i\theta}$), the pressure temperature-gradient correlation ($\phi_{i\theta}$, PTG) and the dissipation ($\varepsilon_{i\theta}$) terms are considered in this study. Among them, the production terms can be calculated if the other second moment quantities, i.e., $\overline{u_i u_j}$, $\overline{u_i\theta}$ and $\overline{\theta^2}$ are given. Thus, the PTG and dissipation are the terms to be modelled in the present work.

The dissipation takes place in the small scale, where the turbulence is isotropic. On the other hand, the expression of the dissipation is not symmetric for the inversion of the space coordinate. Thus, the dissipation of the turbulent heat flux is essentially small. Accordingly, the dissipation is often included in the PTG and is modelled together as follows:

$$\pi_{i\theta} = \phi_{i\theta} - \varepsilon_{i\theta}. \tag{13}$$

This approach is adopted in this study, too.

Firstly, the rapid term is discussed. The rapid term is expressed with introduction of a third-order tensor X_{ijk}:

$$\phi_{i\theta,2} = X_{ijk}\frac{\partial U_j}{\partial x_k} \tag{14}$$

with

$$X_{ijk} = \frac{1}{2\pi}\int \overline{\left(\frac{\partial\theta}{\partial x_i}\right)^{(0)}\left(\frac{\partial u_k}{\partial x_j}\right)'}\frac{\partial V}{\underline{r}}, \tag{15}$$

where the prime $'$ again denotes the position $\underline{r}$ and (0) indicates $r = 0$. In case of the homogeneous turbulence, the tensor X_{ijk} can be further rearranged as

$$X_{ijk} = \frac{1}{2\pi}\int \frac{\partial^2}{\partial r_i \partial r_j}\overline{\theta^{(0)}u_k'}\frac{\partial V}{\underline{r}}. \tag{16}$$

With inspection of Eqs. (15) and (16), the following three restrictions are found to be imposed upon X_{ijk}:

1. the continuity condition of Eq. (15)

$$X_{ikk} = 0 \tag{17}$$

2. the symmetry condition of Eq. (16)

$$X_{ijk} = X_{jik} \tag{18}$$

3. the application of the Green's theorem to Eq. (16) with $i = j$

$$X_{iik} = 2\overline{u_k\theta}. \tag{19}$$

If one assumes here that X_{ijk} is expanded in terms of $\overline{u_i\theta}$ as

$$X_{ijk} = \alpha_1\delta_{ij}\overline{u_k\theta} + \alpha_2\delta_{ik}\overline{u_j\theta} + \alpha_3\delta_{kj}\overline{u_i\theta} \tag{20}$$

then the application of Eqs. (17)–(19) brings a well-known result of

$$\alpha_1 = 0.8 \quad \alpha_2 = \alpha_3 = -0.2. \tag{21}$$

This is the so-called basic model. It is known, however, that this basic model does not always give good results. Common approach (CL, SS) is to add higher order terms of the anisotropy tensor b_{ij}. Preliminary test, however, indicated that the contribution of the higher order terms was small. In the present work, instead, the X_{ijk} is further expanded in terms of the mean-temperature gradient:

$$\begin{aligned} X_{ijk} = {} & \left(\alpha_1\delta_{ij}\overline{u_k\theta} + \alpha_2\delta_{ik}\overline{u_j\theta} + \alpha_3\delta_{kj}\overline{u_i\theta} + \alpha_4 b_{ij}\overline{u_k\theta}\right. \\ & \left. + \alpha_5 b_{ik}\overline{u_j\theta} + \alpha_6 b_{kj}\overline{u_i\theta}\right) \\ & + \left(\beta_1\delta_{ij}\frac{\partial\Theta}{\partial x_k} + \beta_2\delta_{ik}\frac{\partial\Theta}{\partial x_j} + \beta_3\delta_{kj}\frac{\partial\Theta}{\partial x_i}\right. \\ & \left. + \beta_4 b_{ij}\frac{\partial\Theta}{\partial x_k} + \beta_5 b_{ik}\frac{\partial\Theta}{\partial x_j} + \beta_6 b_{kj}\frac{\partial\Theta}{\partial x_i}\right)\overline{q^2}\frac{\overline{q^2}}{\varepsilon} \end{aligned} \tag{22}$$

or equivalently

$$\begin{aligned} X_{ijk} = {} & \left(\alpha_1^*\delta_{ij}\overline{u_k\theta} + \alpha_2^*\delta_{ik}\overline{u_j\theta} + \alpha_3^*\delta_{kj}\overline{u_i\theta}\right) \\ & + \left(\alpha_4\overline{u_i u_j}\overline{u_k\theta} + \alpha_5\overline{u_i u_k}\overline{u_j\theta} + \alpha_6\overline{u_k u_j}\overline{u_i\theta}\right)\frac{1}{\overline{q^2}} \\ & + \left(\beta_1^*\delta_{ij}\frac{\partial\Theta}{\partial x_k} + \beta_2^*\delta_{ik}\frac{\partial\Theta}{\partial x_j} + \beta_3^*\delta_{kj}\frac{\partial\Theta}{\partial x_i}\right)\overline{q^2}\frac{\overline{q^2}}{\varepsilon} \\ & + \left(\beta_4\overline{u_i u_j}\frac{\partial\Theta}{\partial x_k} + \beta_5\overline{u_i u_k}\frac{\partial\Theta}{\partial x_j} + \beta_6\overline{u_k u_j}\frac{\partial\Theta}{\partial x_i}\right)\frac{\overline{q^2}}{\varepsilon}, \end{aligned} \tag{23}$$

where

$$\alpha_1^* = \alpha_1 - (1/3)\alpha_4 \quad \alpha_2^* = \alpha_2 - (1/3)\alpha_5$$
$$\alpha_3^* = \alpha_3 - (1/3)\alpha_6$$

$$\beta_1^* = \beta_1 - (1/3)\beta_4 \quad \beta_2^* = \beta_2 - (1/3)\beta_5$$
$$\beta_3^* = \beta_3 - (1/3)\beta_6.$$

The relations among these coefficients can be obtained from the conditions (17)–(19) for X_{ijk}. Firstly, from the symmetric condition of Eq. (18), it turns out that

$$\alpha_2^* = \alpha_3^*, \quad \alpha_5 = \alpha_6 \tag{24}$$

$$\beta_2^* = \beta_3^*, \quad \beta_5 = \beta_6 \tag{25}$$

Secondly, from the continuity,

$$\begin{aligned} X_{ikk} &= \alpha_1^*\delta_{ik}\overline{u_k\theta} + \alpha_2^*\left(\delta_{ik}\overline{u_k\theta} + \delta_{kk}\overline{u_i\theta}\right) \\ &+ \left\{\alpha_4\overline{u_iu_k}\,\overline{u_k\theta} + \alpha_5\left(\overline{u_iu_k}\,\overline{u_k\theta} + \overline{u_ku_k}\,\overline{u_i\theta}\right)\right\}\frac{1}{\overline{q^2}} \\ &+ \left\{\beta_1^*\delta_{ik}\frac{\partial\Theta}{\partial x_k} + \beta_2^*\left(\delta_{ik}\frac{\partial\Theta}{\partial x_k} + \delta_{kk}\frac{\partial\Theta}{\partial x_i}\right)\right\}\overline{q^2}\frac{\overline{q^2}}{\varepsilon} \\ &+ \left\{\beta_4\overline{u_iu_k}\frac{\partial\Theta}{\partial x_k} + \beta_5\left(\overline{u_iu_k}\frac{\partial\Theta}{\partial x_k} + \overline{u_ku_k}\frac{\partial\Theta}{\partial x_i}\right)\right\}\frac{\overline{q^2}}{\varepsilon} \\ &= 0 \end{aligned} \tag{26}$$

or equivalently

$$\begin{aligned} X_{ikk} &= \left(\alpha_1^* + 4\alpha_2^* + \alpha_5\right)\overline{u_i\theta} + (\alpha_4 + \alpha_5)\frac{\overline{u_iu_k}}{\overline{q^2}}\overline{u_k\theta} \\ &+ \left(\beta_1^* + 4\beta_2^* + \beta_5\right)\overline{q^2}\frac{\overline{q^2}}{\varepsilon}\frac{\partial\Theta}{\partial x_i} \\ &+ \left(\beta_4 + \beta_5\right)\frac{\overline{q^2}}{\varepsilon}\overline{u_iu_k}\frac{\partial\Theta}{\partial x_k} = 0. \end{aligned} \tag{27}$$

Thirdly, from the normalisation

$$\begin{aligned} X_{iik} &= \alpha_1^*\delta_{ii}\overline{u_k\theta} + \alpha_2^*\left(\delta_{ik}\overline{u_i\theta} + \delta_{ki}\overline{u_i\theta}\right) \\ &+ \left\{\alpha_4\overline{u_iu_i}\,\overline{u_k\theta} + \alpha_5\left(\overline{u_iu_k}\,\overline{u_i\theta} + \overline{u_ku_i}\,\overline{u_i\theta}\right)\right\}\frac{1}{\overline{q^2}} \\ &+ \left\{\beta_1^*\delta_{ii}\frac{\partial\Theta}{\partial x_k} + \beta_2^*\left(\delta_{ik}\frac{\partial\Theta}{\partial x_i} + \delta_{ki}\frac{\partial\Theta}{\partial x_i}\right)\right\}\overline{q^2}\frac{\overline{q^2}}{\varepsilon} \\ &+ \left\{\beta_4\overline{u_iu_i}\frac{\partial\Theta}{\partial x_k} + \beta_5\left(\overline{u_iu_k}\frac{\partial\Theta}{\partial x_i} + \overline{u_ku_i}\frac{\partial\Theta}{\partial x_i}\right)\right\}\frac{\overline{q^2}}{\varepsilon} \\ &= 2\overline{u_k\theta}, \end{aligned} \tag{28}$$

which can be rearranged as

$$\begin{aligned} &\left(3\alpha_1^* + 2\alpha_2^* + \alpha_4 - 2\right)\overline{u_i\theta} + 2\alpha_5\frac{\overline{u_iu_k}}{\overline{q^2}}\overline{u_k\theta} \\ &+\left(3\beta_1^* + 2\beta_2^* + \beta_4\right)\overline{q^2}\frac{\overline{q^2}}{\varepsilon}\frac{\partial\Theta}{\partial x_i} + 2\beta_5\frac{\overline{q^2}}{\varepsilon}\overline{u_iu_k}\frac{\partial\Theta}{\partial x_k} \\ &= 0. \end{aligned} \tag{29}$$

If the quantities

$$\overline{u_i\theta},\ \frac{\overline{u_iu_k}}{\overline{q^2}}\overline{u_k\theta},\ \overline{q^2}\frac{\overline{q^2}}{\varepsilon}\frac{\partial\Theta}{\partial x_i},\ \frac{\overline{q^2}}{\varepsilon}\overline{u_iu_k}\frac{\partial\Theta}{\partial x_k} \tag{30}$$

should be independent of each other, the following conditions must be imposed:

$$\begin{array}{ll} \alpha_1^* + 4\alpha_2^* + \alpha_5 = 0 & \alpha_4 + \alpha_5 = 0 \\ \beta_1^* + 4\beta_2^* + \beta_5 = 0 & \beta_4 + \beta_5 = 0 \\ 3\alpha_1^* + 2\alpha_2^* + \alpha_4 = 2 & 2\alpha_5 = 0 \\ 3\beta_1^* + 2\beta_2^* + \beta_4 = 0 & 2\beta_5 = 0. \end{array} \tag{31}$$

It is interesting to note that Eq. (31) reduces to Eqs. (20) and (21). (This was pointed out by one of reviewers.)

The quantities in Eq. (30) are, however, not independent each other. Firstly, a simple eddy diffusivity assumption is often a good approximation. This means the following relation must hold approximately:

$$\begin{aligned} -\overline{u_i\theta} &= \alpha_t\frac{\partial\Theta}{\partial x_i} = \frac{\nu_t}{Pr_t}\frac{\partial\Theta}{\partial x_i} = \frac{C_\mu}{Pr_t}\frac{k^2}{\varepsilon}\frac{\partial\Theta}{\partial x_i} \\ &= \frac{C_\mu}{4Pr_t}\overline{q^2}\frac{\overline{q^2}}{\varepsilon}\frac{\partial\Theta}{\partial x_i}. \end{aligned} \tag{32}$$

Another simple basic representation can be obtained from the balance between the production and slow terms of the PTG:

$$0 = -\overline{u_iu_j}\frac{\partial\Theta}{\partial x_j} + \left(-C_{1\theta}\frac{\varepsilon}{k}\overline{u_i\theta}\right). \tag{33}$$

Thus,

$$-\overline{u_i\theta} = \frac{1}{2C_{1\theta}}\frac{\overline{q^2}}{\varepsilon}\overline{u_iu_j}\frac{\partial\Theta}{\partial x_j}. \tag{34}$$

Eqs. (32) and (33) are quite common assumptions in simple turbulent models. If these two equations are admitted, it means that the following relation is assumed implicitly.

$$\overline{u_i\theta} = -\frac{1}{2C_{1\theta}}\frac{\overline{u_iu_j}}{\overline{q^2}}\overline{q^2}\frac{\overline{q^2}}{\varepsilon}\frac{\partial\Theta}{\partial x_j} = \frac{2Pr_t}{C_{1\theta}C_\mu}\frac{\overline{u_iu_j}}{\overline{q^2}}\overline{u_j\theta} \tag{35}$$

One of the reviewers pointed out that the above relation must hold for the present formulation to be valid. The first impression of Eq. (35) is rather unfamiliar. However, an implication of Eq. (35) is that the turbulent heat flux $\overline{u_j\theta}$ affects $\overline{u_i\theta}$ through the velocity fluctuation correlation $\overline{u_iu_j}$. In other words, if $\overline{u_iu_j}=0$, then $\overline{u_j\theta}$ should cause no effect upon $\overline{u_i\theta}$. This is thus a quite reasonable assumption.

Considering Eqs. (32), (34) and (35) with $C_\mu=0.09$, $Pr_t=0.6$ (for free turbulence) and $C_{1\theta}=3.0$, one obtains

$$\alpha_1^*+4\alpha_2^*+0.225(\alpha_4+\alpha_5)+\alpha_5 - 26.7\left(\beta_1^*+4\beta_2^*+\beta_5\right)-6\left(\beta_4+\beta_5\right)=0 \tag{36}$$

$$3\alpha_1^*+2\alpha_2^*+\alpha_4+0.45\alpha_5 - 26.7\left(3\beta_1^*+2\beta_2^*+\beta_4\right)-12\beta_5=2. \tag{37}$$

Finally, the eight unknown coefficients are determined empirically referring to the various experiments and DNSs with the above correlations (36) and (37) in consideration:

$$\begin{aligned}\alpha_1^*&=0.6, & \alpha_2^*&=-0.1\\ \alpha_4&=1.0, & \alpha_5&=0.45\\ \beta_1^*&=0.02, & \beta_2^*&=-0.005\\ \beta_4&=-0.01, & \beta_5&=0.02\end{aligned} \tag{38}$$

The coefficient of the buoyancy term can be determined from the normalisation condition $Y_{ii}=-\overline{\theta^2}$ as

$$Y_{ii}=-3C_{3\theta}\overline{\theta^2}=-\overline{\theta^2} \tag{39}$$

$$C_{3\theta}=\frac{1}{3}. \tag{40}$$

The slow term is modelled by expanding it in terms of b_{ij}. For the simplicity, only the zeroth- and first-order terms are retained. The second-order term was also examined, but its contribution was sufficiently small. The mean gradient term has been also tried to be included, however, its effect is not consistent in the cases tested in this study. Thus, it is not included in the slow term. Thus, the expression of the slow term becomes

$$\pi_{i\theta,1}=-C_{1\theta,a}\frac{\varepsilon}{k}\overline{u_i\theta}-C'_{1\theta,a}\frac{\varepsilon}{k}b_{ij}\overline{u_j\theta}. \tag{41}$$

Effect of molecular Prandtl Pr was investigated by Iida and Kasagi [8] through DNS of the homogeneous turbulence. Their results indicated a non-negligible effect of Pr on $C_{1\theta,a}$. This is because the dissipation $\varepsilon_{i\theta}$ is included in $C_{1\theta,a}$ and it becomes more significant with decrease of Pr. It is thus found that inclusion of a factor $(1+C/Pr)$ collapses the data as seen in Fig. 1.

Since the decaying turbulence becomes weaker with the elapse of time, the effect of turbulent Reynolds number must be also considered. The examination of DNS data by Ihira and Kawamura [9] indicated that $C_{1\theta,a}$ can be well correlated by introduction of the turbulent Reynolds number (see Fig. 2). Thus, the coefficient $C_{1\theta,a}$ is given as

$$C_{1\theta,a}=3.6\left(1.0+\frac{0.05}{Pr}\right)\left[1.0-0.74\exp\left(-\frac{Re_t}{100}\right)\right]. \tag{42}$$

The second coefficient $C'_{1\theta,a}$ is assumed to be $C'_{1\theta,a}=-2.0$.

Finally, the model equation in the present study becomes

$$\pi_{i\theta,1}=-C_{1\theta,a}\frac{\varepsilon}{k}\overline{u_i\theta}-C'_{1\theta,a}\frac{\varepsilon}{k}b_{ij}\overline{u_j\theta} \tag{43}$$

$$\pi_{i\theta,2}=\alpha_1^*\overline{u_j\theta}\frac{\partial U_i}{\partial x_j}+\alpha_2^*\overline{u_j\theta}\frac{\partial U_j}{\partial x_i}$$

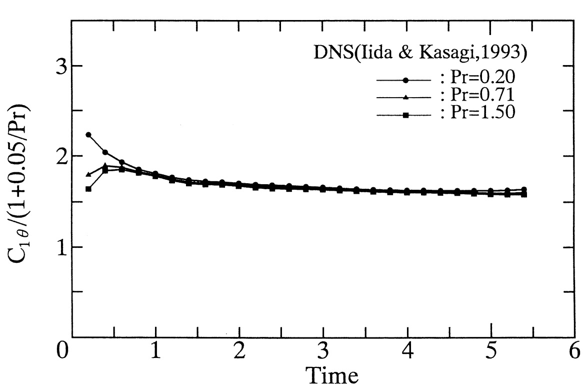

Fig. 1. Coefficient in slow term (effect of Prandtl number).

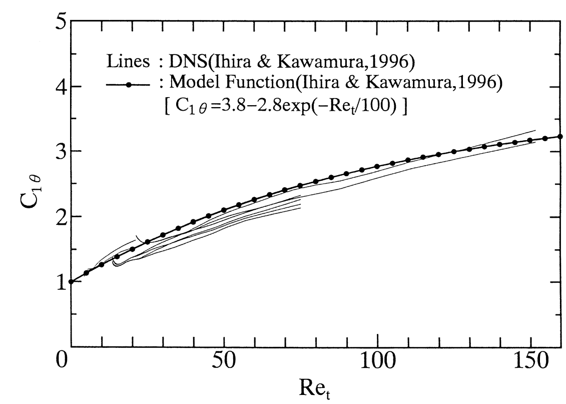

Fig. 2. Coefficient in slow term (effect of turbulent Reynolds number).

$$+\left\{\alpha_4\overline{u_iu_j}\overline{u_k\theta}+\alpha_5\left(\overline{u_iu_k}\overline{u_j\theta}+\overline{u_ku_j}\overline{u_i\theta}\right)\right\}\frac{1}{\overline{q^2}}\frac{\partial U_j}{\partial x_k}$$

$$+\left(\beta_1^*\frac{\partial U_i}{\partial x_k}+\beta_2^*\frac{\partial U_k}{\partial x_i}\right)\overline{q^2}\frac{\overline{q^2}}{\varepsilon}\frac{\partial\Theta}{\partial x_k}$$

$$+\left\{\beta_4\overline{u_iu_j}\frac{\partial\Theta}{\partial x_k}+\beta_5\left(\overline{u_iu_k}\frac{\partial\Theta}{\partial x_j}+\overline{u_ku_j}\frac{\partial\Theta}{\partial x_i}\right)\right\}\frac{\overline{q^2}}{\varepsilon}\frac{\partial U_j}{\partial x_k} \quad (44)$$

$$\pi_{i\theta,3}=-C_{3\theta}(-\beta g_i)\overline{\theta^2} \quad (45)$$

with the coefficients given by Eqs. (38), (40) and (42). The above coefficient has been tested for a moderate range of the Prandtl number as $0.2 < Pr < 1.5$. Extension to a wider range is left for the further investigation.

3. Results

The model derived above is compared with experiment and DNS data. Results given by existing models are also presented for the comparison. Since the present work aims to examine the scalar transport model, the other turbulence quantities such as the Reynolds stresses and the dissipations, are obtained from experiment or simulation; only the turbulent heat flux $\overline{u_i\theta}$ is calculated. The temperature variance $\overline{\theta^2}$ is also calculated in case of the buoyant flow.

Since the homogeneous turbulence is assumed, no diffusion terms appear in the governing equation of the temperature variance.

It can be expressed as

$$\frac{\partial\overline{\theta^2}}{\partial t}=-2\overline{u_j\theta}\frac{\partial\Theta}{\partial x_j}-\varepsilon_\theta, \quad (46)$$

where the dissipation rate ε_θ is again adopted from DNS.

3.1. Anisotropic turbulence without shear and buoyancy

Ihira and Kawamura [10] performed DNS for various combinations of anisotropy *III*, direction of the mean temperature gradient and the Prandtl number. Tested cases are as follows

		$III > 0$	$III < 0$	
$d\Theta/dx_2 = -1$	$Pr = 0.4$	Case P	Case R	(47)
	$Pr = 0.71$	Case Q	Case S	
$d\Theta/dx_3 = -1$	$Pr = 0.4$	Case T	Case U	

Results of comparison are shown in Figs. 3–5. The present model generally gives good predictions. This is mainly because it includes the Prandtl and turbulent Reynolds numbers in the coefficient $C_{1\theta,a}$. Among the other models, GL and JM predict a faster decay. Abe, Kondoh and Nagano [11] (AKN) gives a good agreement in Case U but not in other cases, where it does not satisfy the initial condition. This is because AKN is not a differential equation but an algebraic equation model; thus, the initial condition cannot be specified.

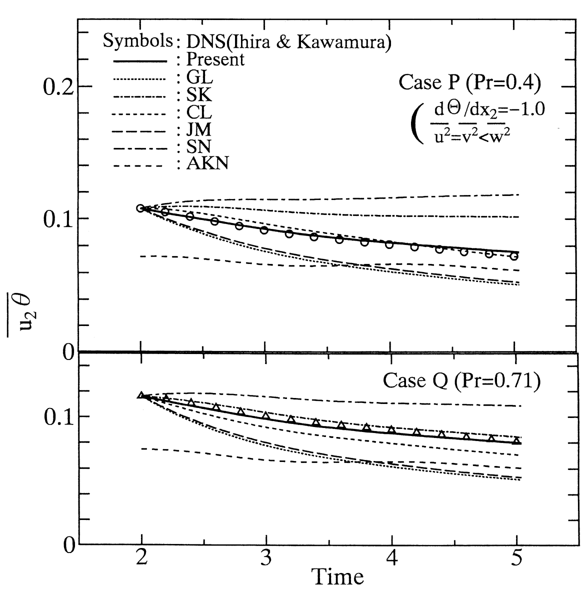

Fig. 3. Variation of turbulent heat flux in an anisotropic turbulence (DNS by Ihira and Kawamura [10]: Cases P and Q).

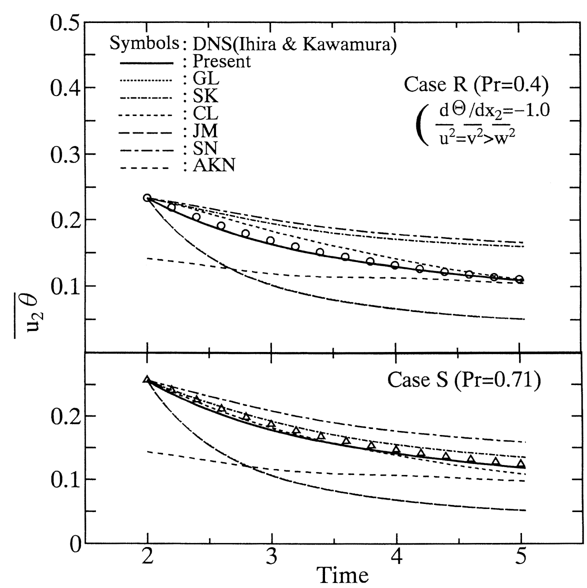

Fig. 4. Variation of turbulent heat flux in an anisotropic turbulence (DNS by Ihira and Kawamura [10]: Cases R and S).

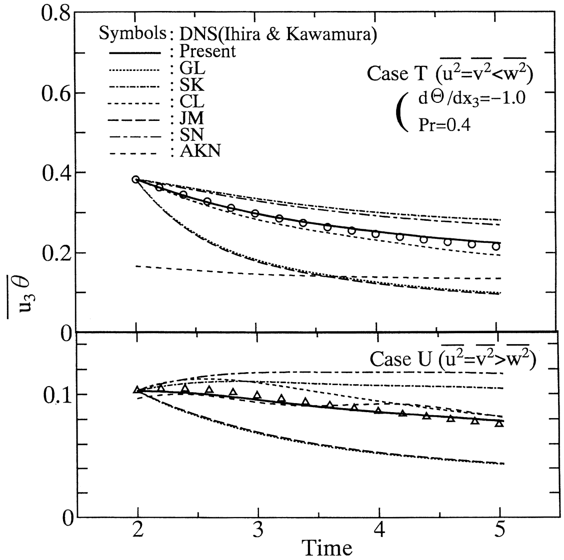

Fig. 5. Variation of turbulent heat flux in an anisotropic turbulence (DNS by Ihira and Kawamura [10]: Cases T and U).

SK predicts well for $Pr = 0.71$, but it gives a slower decay in the other cases. Next comparison is made with DNS by Iida and Kasagi [12]. Four cases are examined

$$\begin{array}{lll} & III > 0 & III < 0 \\ d\Theta/dx_2 = -1 & \text{Case 1} & \text{Case 3} \\ d\Theta/dx_3 = -1 & \text{Case 2} & \text{Case 4} \end{array} \quad (48)$$

The Prandtl number is 0.7 in all cases. The comparison is shown in Figs. 6 and 7. SK and the present model gives a good agreement for all cases. Shimada and Nagano [13] (SN) predicts a slower decay while the other models decay too rapidly.

3.2. Sheared turbulence without buoyancy

Rogers et al. [4] performed DNS for the homogeneous turbulence with mean shear in the direction of x_2 while the mean temperature gradient exists in three different directions, x_1 (Case 1), x_2 (Case 2), and x_3 (Case 3). Results of comparison are shown in Fig. 8. The present model agrees well with their DNS with an only exception of $\overline{u\theta}$ in Case 1. SK gives generally good results; but, as mentioned earlier, it is based tensorially inconsistent expression; so it adopts a different coefficient for $\overline{u\theta}$ and $\overline{v\theta}$. Agreements in other models are generally not so good.

A well-known experiment of this type was made by Tavoularis and Corrsin [14] (TC). The shear and the temperature gradient were imposed in the same direction. Fig. 9 shows the comparison with the model predictions. Most of the models tested give good results. The agreement is better in the heat flux in the direction of mean temperature gradient, while it is slightly worse in the streamwise direction. Similar experiment was made by Maekawa and Kobayashi [15]. The directions of shear and the temperature gradients are same as the former experiment (TC) but their sign is inversed. Comparison with the prediction is given in Fig. 10. The present model gives a good agreement with the experiment. The above comparisons for the sheared tur-

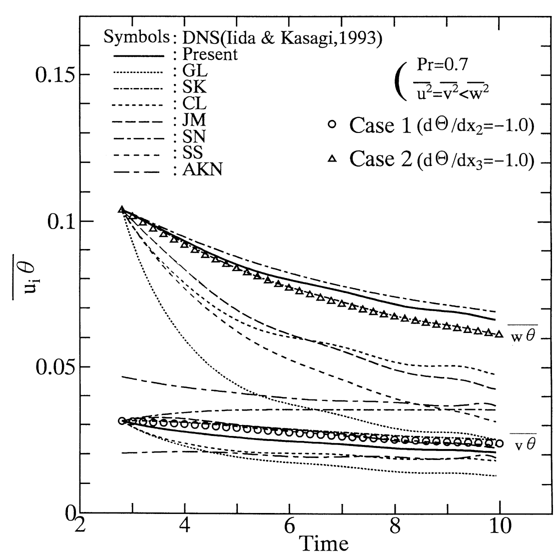

Fig. 6. Variation of turbulent heat flux in an anisotropic turbulence (DNS by Iida and Kasagi [12]: Cases 1 and 2).

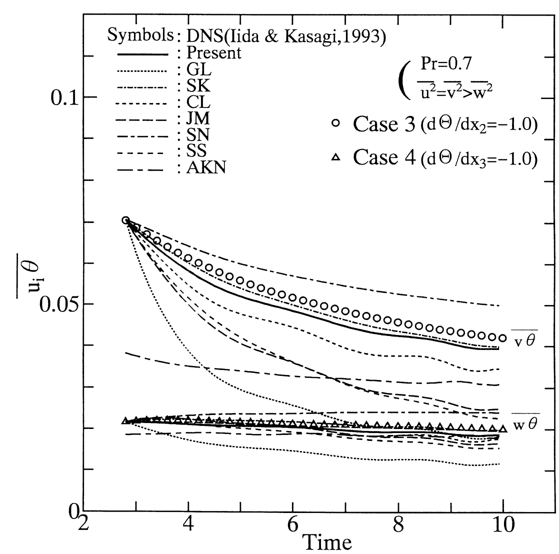

Fig. 7. Variation of turbulent heat flux in an anisotropic turbulence (DNS by Iida and Kasagi [12]: Cases 3 and 4).

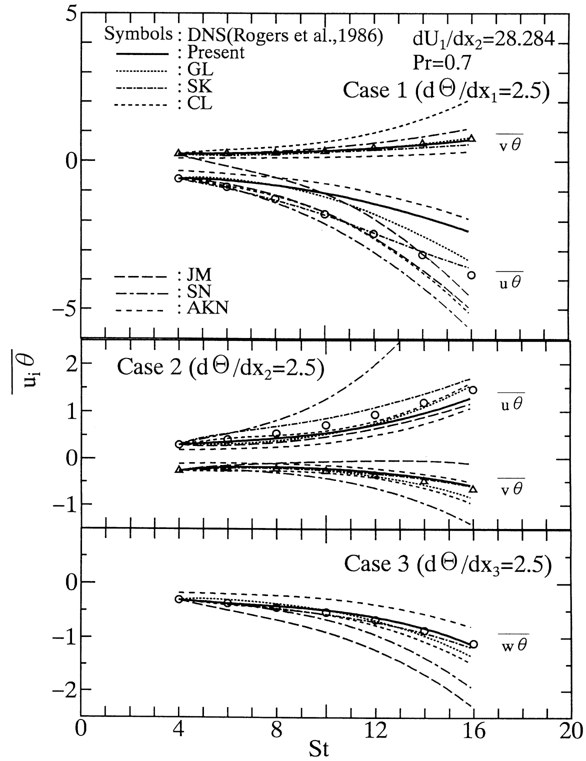

Fig. 8. Variation of turbulent heat flux in a sheared turbulence (DNS by Rogers et al. [4]).

bulence indicates that the prediction by this model as well as by the other ones agrees better with the experiment than with the DNS data. In the DNS, the shear is imposed suddenly; thus the turbulence spectrum develops with the elapse of time. This situation is difficult to be predicted by this kind of turbulence model, because it deals with only the quantities integrated with respect to the spectrum.

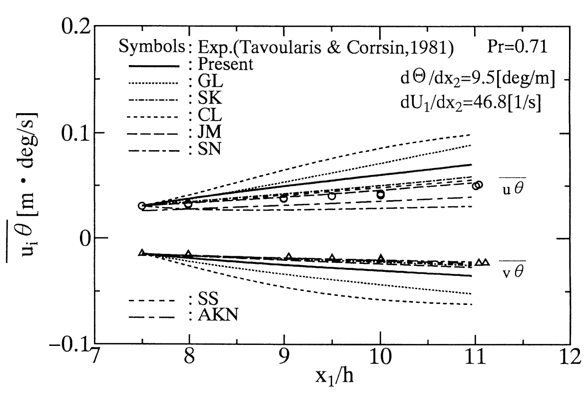

Fig. 9. Variation of turbulent heat flux in a sheared turbulence (experiment by Tavoularis and Corrsin [14]).

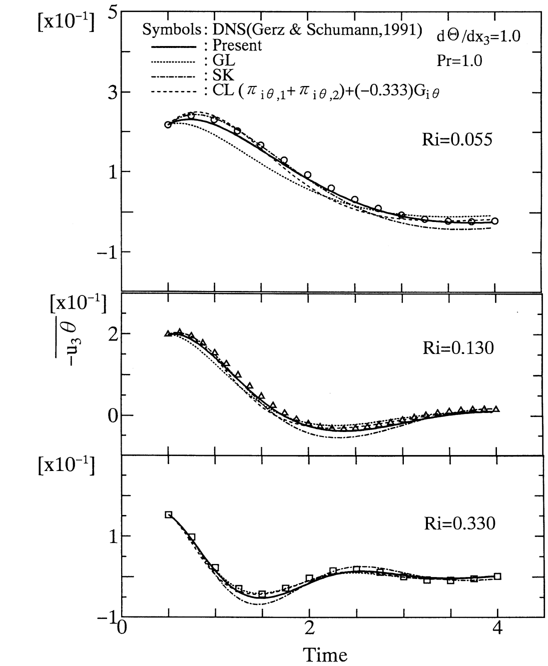

Fig. 11. Variation of turbulent heat flux in a stably stratified turbulence (DNS by Gerz and Schumann [16]).

3.3. Stratified turbulence without mean shear

Gerz and Schumann [16] performed DNS of the stably stratified turbulence without mean shear. The positive temperature gradient is imposed in the direction of x_3 with the gravity vector (0, 0, $-g$). The results are

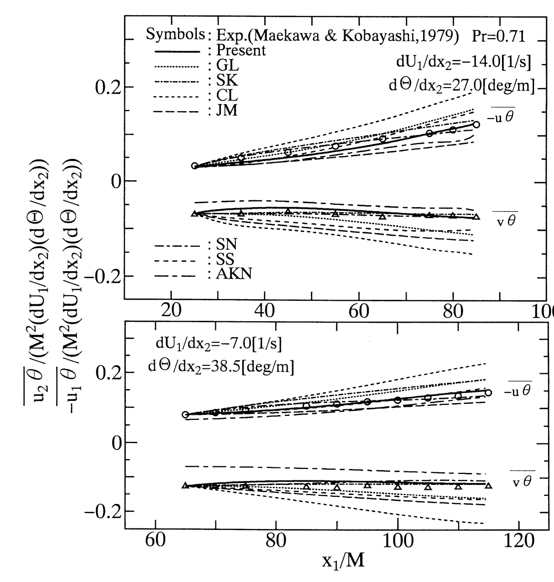

Fig. 10. Variation of turbulent heat flux in a sheared turbulence (experiment by Maekawa and Kobayashi [15]).

shown in Fig. 11, the vertical heat flux becomes oscillatory with increase of the Richardson number

$$Ri = g\frac{\partial\rho/\partial x_3}{\rho(\partial U_1/\partial x_3)^2}. \tag{49}$$

This tendency is well predicted by almost all the models tested. The heat flux becomes negative in some period; this is the counter gradient heat flux (CGHF). This is also well reproduced by the model predictions.

3.4. Stratified turbulence with mean shear

Gerz et al. made DNS [17] and LES [18] of the stratified and sheared turbulence. Comparison is made in Fig. 12. In this figure, the gradient Richardson number is defined as

$$Ri_g = \beta g\frac{\partial\Theta/\partial x_3}{(\partial U_1/\partial x_3)^2}. \tag{50}$$

In case of a low Ri_g ($Ri_g = 0.13$), it is interesting to note that the heat flux stays in a quasi-stationary level. This indicates that the decay rate of turbulence is balanced with the production by the mean shear. This behaviour is well predicted by the present model. GL also agrees well the DNS. The CGHF occurs in case of the high Ri_g; and it is well captured by the models tested.

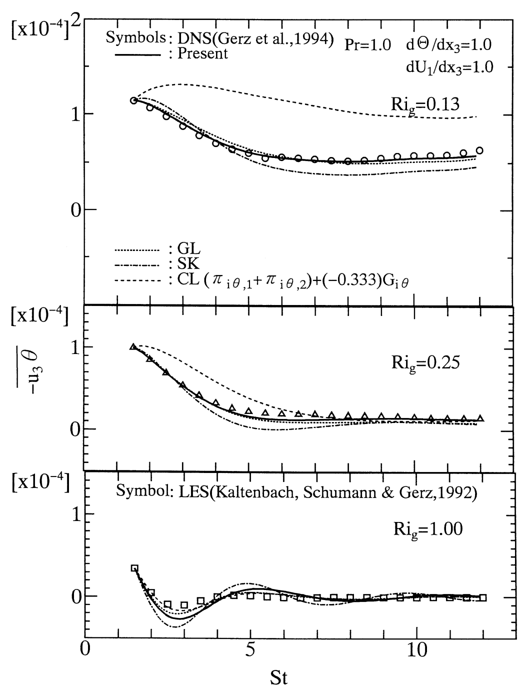

Fig. 12. Variation of turbulent heat flux in a stably stratified sheared turbulence (DNS by Gerz et al. [17] and LES by Kaltenbach et al. [18]).

4. Conclusion

A new model for turbulent heat flux transport was developed with the effect of the mean temperature gradient directly taken into account. The derivation of the rapid term was reexamined including the mean temperature gradient explicitly. The linearity in the exact transport equation for the turbulent heat flux was retained in the modelled equation. The proposed model was tested in comparison with the DNSs and experiments of homogenous turbulence with and without the mean shear rate and the stable stratification. The new model gave good agreement with the DNS and experimental data.

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